

Problem 2.19

[Difficulty: 3]

2.19 Consider the flow described by the velocity field $\vec{V} = A(1 + Bt)\hat{i} + Cty\hat{j}$, with $A = 1 \text{ m/s}$, $B = 1 \text{ s}^{-1}$, and $C = 1 \text{ s}^{-2}$. Coordinates are measured in meters. Plot the pathline traced out by the particle that passes through the point $(1, 1)$ at time $t = 0$. Compare with the streamlines plotted through the same point at the instants $t = 0, 1$, and 2 s .

Given: Velocity field

Find: Plot of pathline traced out by particle that passes through point $(1, 1)$ at $t = 0$; compare to streamlines through same point at the instants $t = 0, 1$ and 2 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = A \cdot (1 + B \cdot t)$ $A = 1 \cdot \frac{\text{m}}{\text{s}}$ $B = 1 \cdot \frac{1}{\text{s}}$ $v_p = \frac{dy}{dt} = C \cdot t \cdot y$ $C = 1 \cdot \frac{1}{\text{s}^2}$

So, separating variables $dx = A \cdot (1 + B \cdot t) \cdot dt$ $\frac{dy}{y} = C \cdot t \cdot dt$

Integrating $x = A \cdot \left(t + B \cdot \frac{t^2}{2} \right) + C_1$ $\ln(y) = \frac{1}{2} \cdot C \cdot t^2 + C_2$
 $y = e^{\frac{1}{2} \cdot C \cdot t^2 + C_2} = e^{C_2} \cdot e^{\frac{1}{2} \cdot C \cdot t^2} = c_2 \cdot e^{\frac{1}{2} \cdot C \cdot t^2}$

The pathlines are $x = A \cdot \left(t + B \cdot \frac{t^2}{2} \right) + C_1$ $y = c_2 \cdot e^{\frac{1}{2} \cdot C \cdot t^2}$

Using given data $x = A \cdot \left(t + B \cdot \frac{t^2}{2} \right) + 1$ $y = e^{\frac{1}{2} \cdot C \cdot t^2}$

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{C \cdot y \cdot t}{A \cdot (1 + B \cdot t)}$

So, separating variables $(1 + B \cdot t) \cdot \frac{dy}{y} = \frac{C}{A} \cdot t \cdot dx$ which we can integrate for any given t (t is treated as a constant)

Integrating $(1 + B \cdot t) \cdot \ln(y) = \frac{C}{A} \cdot t \cdot x + c$

The solution is $y^{1+B \cdot t} = \frac{C}{A} \cdot t \cdot x + \text{const}$ $y = \left(\frac{C}{A} \cdot t \cdot x + \text{const} \right)^{\frac{1}{(1+B \cdot t)}}$

For particles at (1,1) at t = 0, 1, and 2s, using A, B, and C data:

$$y = 1$$

$$y = x^{\frac{1}{2}}$$

$$y = (2 \cdot x - 1)^{\frac{1}{3}}$$

